CERN-PH-TH/2009-080

Black hole entropy, flat directions and higher derivatives

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Abstract

Higher order derivative corrections to the Einstein–Maxwell action are considered and an explicit form is found for the corrections to the entropy of extremal black holes. We speculate on the properties of these corrections from the point of view of small black holes and in the case when the classical black hole potential exhibits flat directions. A particular attention is paid to the issue of stability of several solutions, including large and small black holes by using properties of the Hessian matrix of the effective black hole potential. This is done by using a model independent expression for such matrix derived within the entropy function formalism.

1 Introduction

Black hole physics provides us with a vast variety of phenomena for testing underlying ideas of theoretical physics. This explains the constant attention to this topical issue, as well as the burst of interest any time a new concept emerges in this area of research. Superstring/M-theory has enriched physics with new ideas and currently is the main subject of interest in mathematical physics. It allowed to intertwine different areas of theoretical physics, e.g. gravity and conformal field theories.

There is a well known correspondence between black hole mechanics and thermodynamics [1] that relates geometrical characteristics of a black hole to thermodynamical ones. One of them, the subject of our interest here, – entropy – can be calculated using the Wald formula [2]. To be able to interpret this quantity as a genuine entropy, it should be confirmed by statistical physics calculations. This was accomplished by performing counting of the microstates [3, 4] in the "classical" limit. At this point, there arises a question of comparison between microstate and "macrostate" entropies beyond the classical approximation.

For extremal black holes the method of calculation of macrostate entropy was proposed in [5]. It relies on the classical Einstein-Maxwell action and reduces the problem to finding a black hole potential which encodes all the information needed to obtain the entropy, which is given by the value of the black hole potential at its critical point. This method allows one to trace back the evolution of the scalar fields in the whole space, but might become difficult when considering higher order corrections which originate from the string coupling and the α' expansion of the superstring action.

For this purpose another approach was proposed in [6]. It deals with horizon values of the fields present in the theory by means of the so-called entropy function.

Taking into account terms coming from the superstring action (such as the Chern–Simons term) allows one to achieve a matching between black holes microstate and "macrostate" entropies [7].

Higher order corrections might be of interest from the classical point of view due to the following problems:

1. small black hole – black hole with vanishing classical entropy and non-vanishing values of mass. Once classically one obtains a zero value of the entropy, one naturally poses the question what happens when taking into consideration higher order corrections as well.

2. flat directions of the black hole potential and stability of the critical points. The presence of flat directions reflects the symmetry of the scalar manifold. Therefore it is interesting to know how the symmetry gets modified in the presence of higher order corrections and, as a consequence, how flat directions get distorted and whether the resulting critical point is stable or not.

The problem of small black holes is studied from different points of view [8].

In order to investigate the influence of higher order corrections on the solutions to the attractor mechanism equations [9], we make use of the black hole potential approach and the entropy function formalism. We established a direct relation between these methods and found how the black hole potential is related to the entropy function. In addition, we derive a formula (10) for the Hessian matrix of the black hole potential completely within the entropy function formalism. A distinctive feature of this formula is that it can be easily used even

if deriving the expression of the black hole potential from the entropy function proves to be problematic.

Previously, in the entropy function formalism not much attention was paid to the stability issues, due to the fact that a truncation over the "problematic" sectors (those where the flat directions reside, e.g. the axionic one in the *stu* model) was always considered. Having at disposal the formula for the Hessian matrix allows us to study the stability of the complete, non truncated version of the theory.

In the present work, we consider the most general case of higher order derivative corrections to the Einstein-Maxwell action involving both Riemann curvature tensor $R_{\mu\nu\rho\sigma}$ and electromagnetic field strength tensor $F^{\Lambda}_{\mu\nu}$ in any Lorentz covariant combinations. From dimensional analysis it follows that the n^{th} order correction should have the form $(R_{\mu\nu\rho\sigma})^m(F^{\text{L}}_{\mu\nu})^{2(n-m+1)}$ with $m = 0, \ldots, n+1$. The indices are supposed to be contracted in all possible ways by means of a metric $g_{\mu\nu}$. The number of such terms grows very fast with respect to n, but in the entropy function formalism they all lead to a much smaller number of independent combinations. In this way we succeed in finding the corrections to the black hole entropy induced by the higher order derivative corrections to action.

As it is known, usually, in order to calculate up to n^{th} order the value of a function at a critical point, one should know the solution to the criticality condition up to $(n-1)^{\text{th}}$ order. Examining the form of the corrections to the entropy, one might easily observe that there appears a problem starting from the first order. Namely, the first order correction to the entropy is supposed to be determined by the classical solution to the attractor mechanism equations. Once the classical black hole potential possesses flat directions, not all moduli are determined by the attractor mechanism equations, though this ambiguity does not affect the classical value of the entropy. But in the first order this ambiguity shows up: the entropy becomes dependent on the undetermined moduli. Evidently this fact contradicts the second law of black hole thermodynamics and the attractor mechanism paradigm. Nevertheless, a thorough analysis of the perturbative corrections shows that there exists a sort of feedback from the first order solutions on the classical ones. In general, in order to calculate the entropy up to n^{th} order, one should have the solution up to n^{th} order. As it was mentioned above, n^{th} order solution does not show up in the expression for the entropy (at least to n^{th} order), but it might not exist in the presence of flat directions. This is related to the fact that, in order to find the n^{th} order solution, one has to solve a system of linear non-homogeneous equations with vanishing determinant. This system contains as parameters moduli which are not defined in the previous orders. Some of these parameters get fixed when requiring the system of equations to be consistent. In this way the equations to n^{th} order fix the solutions in previous orders. Exactly for this reason the attractor mechanism remains valid in the presence of higher order derivative corrections, as well.

We considered in detail an example of a *stu* dyonic black hole in heterotic string theory compactified on T^6 or $K3 \times T^2$. In addition to previously known solutions [7, 10], we found two new solutions. They are both non-BPS and one of them, moreover, corresponds to a state with vanishing central charge Z [11]. In the classical limit the latter solution turns out to be stable. Therefore we dwell mainly on the stability of the non-BPS $Z \neq 0$ solution, which classically has two vanishing and four positive eigenvalues. We find that, when the corrections are turned on, one of the previously vanishing eigenvalues remains zero, while the other becomes positive.

We studied as well the small black hole limit and found that the two new solutions mentioned above do not allow for such a limit, while the previously known ones do. Therefore we studied

the stability of these solutions, as well, and found that for both of them in the small black hole limit the Hessian matrix has always at least one negative eigenvalue.

The paper is organized as follows. In section 2 we establish a relation between the two methods for calculating the black hole entropy (6). The main result of this section is the formula (10) yielding the Hessian matrix expressed in terms of the entropy function. In section 3 we introduce the higher order corrections to the action, discuss their explicit form and derive the expressions for the black hole effective potential (17) and entropy (26), up to the second order. We speculate on properties of the corrections to the entropy both in the case of small black holes and when the classical black hole potential has flat directions. In section 4 we revisit a well known example of heterotic string theory compactified on T^6 or $K3 \times T^2$ without neglecting axions. We demonstrate here that the corrections to the entropy (39) do not depend on values of the scalars at infinity. We derive two new non-BPS solutions and investigate their stability, as well as the stability of the previously known ones. A special attention is paid to small black holes and the issue of their stability.

2 Basics

Since the paper [5], it has been known how to calculate the entropy of a wide class of extremal black holes. There an Einstein–Maxwell model coupled to scalar fields was considered

$$S = \int d^4x\sqrt{-g}\,\Big[-\frac{R}{2} + \frac{1}{2}G_{ab}(\phi)\, g^{\mu\nu}\,\partial_\mu\phi^a\partial_\nu\phi^b - \frac{1}{4}\mu_{\Lambda\Sigma}(\phi)F^{\Lambda}_{\mu\nu}F^{\Sigma\mu\nu} - \frac{1}{4}\nu_{\Lambda\Sigma}(\phi)F^{\Lambda}_{\mu\nu} * F^{\Sigma\mu\nu}\Big] \quad (1)$$

and it was shown that the entropy of a black hole in a static, spherically symmetric and asymptotically flat space-time is completely determined by the so-called *black hole potential*, which for the model (1) has the form

$$V_{BH}(\phi) = -\frac{1}{2}(p^{\Lambda},\, q_{\Lambda}) \begin{pmatrix} \mu_{\Lambda\Sigma} + \nu_{\Lambda\Gamma}\mu^{\Gamma\Pi}\nu_{\Pi\Sigma} & \nu_{\Lambda\Gamma}\mu^{\Gamma\Sigma} \\ \mu^{\Lambda\Gamma}\nu_{\Gamma\Sigma} & \mu^{\Lambda\Sigma} \end{pmatrix} \begin{pmatrix} p^{\Sigma} \\ q_{\Sigma} \end{pmatrix}. \quad (2)$$

The entropy within this framework is given by the value of the black hole potential at the horizon

$$S = \pi V_{BH}(\phi_h), \quad (3)$$

where the moduli take critical values obtained from

$$\left.\frac{\partial V_{BH}}{\partial \phi^a}\right|_{\phi=\phi_h} = 0. \quad (4)$$

This approach naturally allows us to examine the stability (which guarantees that the expression in (3) refers to the entropy of a physical object) of the critical points by checking the positive definiteness of the Hessian matrix

$$H_{ab} = \left.\frac{\partial^2 V_{BH}}{\partial\phi^a\partial\phi^b}\right|_{\phi=\phi_h},$$

and it gives us the possibility to establish the presence of flat directions of the black hole potential. Classically, the flat directions are not fraught with any problem. Namely, the criticality condition of V_{BH} might not fix uniquely the values of the scalar fields, and some

combinations of the fields might remain free, while the value of the entropy does not depend on these combinations. The problem might be hidden in quantum correction terms which, in general, might destabilize the flat direction (transforming it into a saddle critical point).

The presence of the additional terms – whether of quantum or classical origin – in the action (1) leads to a modification of the black hole potential; there arises the so-called effective black hole V_{eff} potential, whose extreme value is equal to the entropy of the black hole and whose critical points give the values of the scalar fields at the black hole horizon. For the case of model (1) the effective potential V_{eff} is equal to the black hole one V_{BH}.

An alternative approach – the entropy function formalism – for calculating the entropy of a black hole with higher order derivative corrections was proposed in [6, 12]. It may also be applied to calculate the entropy of a large class of systems with additional degrees of freedom, e.g. rotating black holes, black strings, multi-center black hole configurations. Since we are also interested in stability issues, we first relate this approach to the black hole potential one and derive the concept of the Hessian matrix in the entropy function formalism. Let us recall that the latter is based on putting the theory on the near-horizon background geometry and constructing the so-called *entropy function* which can be easily written down once the Lagrangian is known

$$\mathcal{E}(E^I, \phi^a) = 2\pi \left[e^{\scriptscriptstyle L} q_{\scriptscriptstyle L} - \int d\theta d\varphi \sqrt{-g}\mathcal{L} \right],$$

where E^I stands for all fields but the moduli. In this formalism the black hole entropy is given by the value of the entropy function at the horizon. The horizon values E^I_h and ϕ^a_h of all fields are determined by resolving all criticality conditions simultaneously

$$\left.\frac{\partial \mathcal{E}}{\partial E^I}\right|_{\substack{E=E_h \\ \phi=\phi_h}} = 0, \quad \text{(a)} \qquad\qquad \left.\frac{\partial \mathcal{E}}{\partial \phi^a}\right|_{\substack{E=E_h \\ \phi=\phi_h}} = 0. \quad \text{(b)} \tag{5}$$

Since the entropy function formalism is more general, it encompasses the black hole potential approach. One can obtain the black hole potential from the entropy function as follows:

$$V_{BH}(\phi) = \pi^{-1}\, \mathcal{E}(E^I(\phi), \phi) \tag{6}$$

where $E^I(\phi)$ denotes the solution to the equations $\partial\mathcal{E}/\partial E^I = 0$ in terms of the moduli. Of course, one can find the black hole potential, provided it is possible to resolve this equation explicitly. For example, in order to reproduce the black hole potential (2), one should fix the near horizon geometry as $AdS_2 \times S^2$

$$ds^2 = -v_1 \left(r^2 dt^2 - \frac{dr^2}{r^2} \right) + v_2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right), \tag{7}$$

resolve Bianchi identities for the electromagnetic fields

$$F^{\scriptscriptstyle L}_{01} = -e^{\scriptscriptstyle L}(r), \qquad F^{\scriptscriptstyle L}_{23} = p^{\scriptscriptstyle L} \sin\theta \tag{8}$$

and substitute the explicit dependence of $E^I = (e^{\scriptscriptstyle L}, v_1, v_2)$ on ϕ^a

$$e^{\scriptscriptstyle L} = -\mu^{\scriptscriptstyle L\Sigma}\left(q_{\scriptscriptstyle \Sigma} - \nu_{\scriptscriptstyle \Sigma\Omega} p^{\scriptscriptstyle \Omega} \right), \quad v_1 = v_2 = -\frac{1}{2}\left[\mu_{\scriptscriptstyle L\Sigma} p^{\scriptscriptstyle L} p^{\scriptscriptstyle \Sigma} + \mu^{\scriptscriptstyle L\Sigma}\left(q_{\scriptscriptstyle L} - \nu_{\scriptscriptstyle LL'} p^{\scriptscriptstyle L'} \right)\left(q_{\scriptscriptstyle \Sigma} - \nu_{\scriptscriptstyle \Sigma\Sigma'} p^{\scriptscriptstyle \Sigma'} \right) \right] \tag{9}$$

back into the entropy function.

In many cases one cannot resolve the criticality conditions (5a) and hence derive the black hole potential, therefore the entropy function formalism is more general. In order to make it self-contained one should be able to answer the question about the stability, that is to calculate the Hessian matrix in terms of the entropy function only. It becomes important when one cannot deduce the explicit dependence of E^I on ϕ^a.

From the formula (6) it follows that the Hessian matrix is equal to

$$H_{ab} = \frac{1}{\pi}\left[\frac{\partial^2\mathcal{E}}{\partial E^I\partial E^J}\frac{\partial E^I}{\partial\phi^a}\frac{\partial E^J}{\partial\phi^b} + \frac{\partial^2\mathcal{E}}{\partial E^I\partial\phi^a}\frac{\partial E^I}{\partial\phi^b} + \frac{\partial^2\mathcal{E}}{\partial E^I\partial\phi^b}\frac{\partial E^I}{\partial\phi^a} + \frac{\partial^2\mathcal{E}}{\partial\phi^a\partial\phi^b}\right]_{\substack{E=E_h\\ \phi=\phi_h}} .$$

Supposing that one does not have the explicit form of $E^I(\phi)$, let us try to express $\partial E^I/\partial\phi^a$ through the entropy function. For this purpose it is sufficient to differentiate $\partial\mathcal{E}/\partial E^I = 0$ with respect to ϕ^a, considering E^I as a function of ϕ^a:

$$\frac{\partial^2\mathcal{E}}{\partial E^I\partial E^J}\frac{\partial E^J}{\partial\phi^a} + \frac{\partial^2\mathcal{E}}{\partial E^I\partial\phi^a} = 0 \quad\Rightarrow\quad \frac{\partial E^I}{\partial\phi^a} = -\left(\frac{\partial^2\mathcal{E}}{\partial E^I\partial E^J}\right)^{-1}\frac{\partial^2\mathcal{E}}{\partial E^J\partial\phi^a}$$

which yields the following expression for the Hessian matrix:

$$H_{ab} = \frac{1}{\pi}\left[\frac{\partial^2\mathcal{E}}{\partial\phi^a\partial\phi^b} - \left(\frac{\partial^2\mathcal{E}}{\partial E^I\partial E^J}\right)^{-1}\frac{\partial^2\mathcal{E}}{\partial E^I\partial\phi^a}\frac{\partial^2\mathcal{E}}{\partial E^J\partial\phi^b}\right]_{\substack{E=E_h\\ \phi=\phi_h}} . \tag{10}$$

Notice, that this expression is quite simple to deal with. It requires the knowledge of the derivatives of the entropy function which are easily calculable. Amusingly enough, for studying the stability of the solutions, one does not have to be able to resolve equations $\partial\mathcal{E}/\partial E^I = 0$ in terms of the moduli in order to reconstruct the black hole potential. This issue becomes important when considering higher order derivative corrections.

3 Higher order corrections

Now let us expand the considerations we made above by adding higher order derivative terms to the action (1):

$$\begin{aligned} S = \int d^4x\sqrt{-g}\,\Big[-\frac{R}{2} + \frac{1}{2}G_{ab}(\phi)\,g^{\mu\nu}\,\partial_\mu\phi^a\partial_\nu\phi^b - \frac{1}{4}\mu_{\Lambda\Sigma}(\phi)F^{\Lambda}_{\mu\nu}F^{\Sigma\mu\nu} \\ -\frac{1}{4}\nu_{\Lambda\Sigma}(\phi)F^{\Lambda}_{\mu\nu} * F^{\Sigma\mu\nu}\Big] + \int d^4x\sqrt{-g}\,\mathcal{L}_H(R_{\mu\nu\lambda\sigma}, g^{\mu\nu}, F^{\Lambda}_{\mu\nu}, \phi) \end{aligned} \tag{1$'$}$$

The last term corresponds to higher order derivative corrections coming, for example, from the α'-expansion of the heterotic string action [13]; its form will be specified later.

The entropy function corresponding to the model (1$'$) acquires an additional term

$$\mathcal{E}(e,v,\phi) = 2\pi\left[e^{\Lambda}q_{\Lambda} + v_2 - v_1 - \frac{1}{2}\mu_{\Lambda\Sigma}\left(\frac{v_1}{v_2}p^{\Lambda}p^{\Sigma} - \frac{v_2}{v_1}e^{\Lambda}e^{\Sigma}\right) - \nu_{\Lambda\Sigma}e^{\Lambda}p^{\Sigma}\right] + \mathcal{E}_H \tag{11}$$

which is defined as an integral calculated on $AdS_2\times S^2$ background (7) with resolved Bianchi identities (8)

$$\mathcal{E}_H(v_1, v_2, p, e, \phi) = -\frac{1}{4}\int_{S^2} d\theta d\varphi\sqrt{-g}\,\mathcal{L}_H. \tag{12}$$

Since the contribution from the term $\mathcal{E}_H$ is supposed to come from the perturbative expansion of the superstring action, we present it in the following form:

$$\mathcal{E}_H = \alpha'\mathcal{E}_1 + \alpha'^2\mathcal{E}_2 + O(\alpha'^3). \tag{13}$$

Our goal is to construct the black hole potential corresponding to the action (1′). In what follows we reserve the term "black hole potential" for a potential corresponding to the classical action (1), and the term "effective black hole potential" – to the action (1′) with higher order derivative corrections.

As it was demonstrated in the previous section, to obtain the effective potential one should eliminate in the entropy function all fields except the moduli, using their equations of motion. For this purpose we represent the entropy function in the form

$$\mathcal{E}(e, v, \phi) = \mathcal{E}_0(e, v, \phi) + \alpha'\mathcal{E}_1(e, v, \phi) + \alpha'^2\mathcal{E}_2(e, v, \phi) + O(\alpha'^3) \tag{14}$$

and for simplicity introduce again a notation $E^I = (v_1, v_2, e^{\text{L}})$ for the "superfluous" fields. It can be easily shown that the effective potential, up to the second order in α', is then given by

$$V_{eff}(\phi) = \frac{1}{\pi}\left[\mathcal{E}(E^I, \phi^a) - \frac{\alpha'^2}{2}\,(H_{IJ})^{-1}\frac{\partial\mathcal{E}_1}{\partial E^I}\frac{\partial\mathcal{E}_1}{\partial E^J}\right]_{E=E_0} + O(\alpha'^3), \quad H_{IJ} = \frac{\partial^2\mathcal{E}_0}{\partial E^I\partial E^J} \tag{15}$$

where E_0 stands for the "classical" solutions (9) for the fields $v_{1,2}$ and e^{L} in terms of the moduli. The matrix H^{-1} has a following block form:

$$H^{-1} = -\frac{1}{2\pi}\begin{pmatrix} 2V_{BH} & V_{BH} & e^{\text{L}} \\ V_{BH} & 0 & e^{\text{L}} \\ e^{\Sigma} & e^{\Sigma} & -\mu^{\text{L}\Sigma} \end{pmatrix} \tag{16}$$

with V_{BH} and e^{L} given in eqs. (2) and (9). Therefore we may represent the effective potential in the form

$$\begin{aligned} V_{eff}(\phi) &= V_{BH}(\phi) + \frac{\alpha'}{\pi}\,\mathcal{E}_1(E_0(\phi), \phi) + \frac{\alpha'^2}{\pi}\,\mathcal{E}_2(E_0(\phi), \phi) \\ &\quad - \frac{\alpha'^2}{4\pi^2}\left[\mu^{\text{L}\Sigma}\frac{\partial\mathcal{E}_1}{\partial e^{\text{L}}}\frac{\partial\mathcal{E}_1}{\partial e^{\Sigma}} - \frac{2}{V_{BH}}\left(v_1\frac{\partial\mathcal{E}_1}{\partial v_1} + e^{\text{L}}\frac{\partial\mathcal{E}_1}{\partial e^{\text{L}}}\right)\left(v_1\frac{\partial\mathcal{E}_1}{\partial v_1} + v_2\frac{\partial\mathcal{E}_1}{\partial v_2}\right)\right] + O(\alpha'^3) \end{aligned} \tag{17}$$

with the right hand side, obviously, being calculated on the classical solution (9).

3.1 Explicit form of the corrections

Now we try to make our considerations more specific by defining the form of the corrections. Although we are interested only up to α'^2 order corrections to the effective potential, the formulae of this section might be easily written down for the corrections of any order.

To the α'^1 order, the possible corrections allowed by dimensinal analysis have the following schematic structure:

$$\mathcal{L}_H^{(1)} \quad\sim\quad (R_{\mu\nu\rho\sigma})^2 \quad+\quad R_{\mu\nu\rho\sigma}(F_{\mu\nu})^2 \quad+\quad (F_{\mu\nu})^4,$$

where the Lorentz indices are supposed to be contracted with a proper number of inverse metric $g^{\mu\nu}$ in all possible ways. This expression induces the following form of the first order correction to the entropy function:

$$\mathcal{E}_1 = \mathcal{E}_{R^2} + \mathcal{E}_{RF^2} + \mathcal{E}_{F^4}. \tag{18}$$

Calculated on the $AdS_2 \times S^2$ background (7) and with the field strength given by eqs. (8), the terms composing $\mathcal{E}_1$ are then given by

$$
\begin{aligned}
\mathcal{E}_{R^2} &\sim v_1 v_2 \left[\frac{\alpha^{(1)}}{v_1^2} + \frac{\alpha^{(2)}}{v_1 v_2} + \frac{\alpha^{(3)}}{v_2^2} \right] , \\
\mathcal{E}_{RF^2} &\sim v_1 v_2 \left[\left(\frac{\alpha^{(4)}_{\text{Ł}\Sigma}}{v_1} + \frac{\alpha^{(5)}_{\text{Ł}\Sigma}}{v_2} \right) \frac{e^{\text{Ł}} e^{\Sigma}}{v_1^2} + \left(\frac{\alpha^{(6)}_{\text{Ł}\Sigma}}{v_1} + \frac{\alpha^{(7)}_{\text{Ł}\Sigma}}{v_2} \right) \frac{e^{\text{Ł}} p^{\Sigma}}{v_1 v_2} + \left(\frac{\alpha^{(8)}_{\text{Ł}\Sigma}}{v_1} + \frac{\alpha^{(9)}_{\text{Ł}\Sigma}}{v_2} \right) \frac{p^{\text{Ł}} p^{\Sigma}}{v_2^2} \right] , \\
\mathcal{E}_{F^4} &\sim v_1 v_2 \left[\alpha^{(10)}_{\text{Ł}\Sigma\Pi\Omega} \frac{e^{\text{Ł}} e^{\Sigma} e^{\Pi} e^{\Omega}}{v_1^4} + \alpha^{(11)}_{\text{Ł}\Sigma\Pi\Omega} \frac{e^{\text{Ł}} e^{\Sigma} e^{\Pi} p^{\Omega}}{v_1^3 v_2} + \ldots + \alpha^{(14)}_{\text{Ł}\Sigma\Pi\Omega} \frac{p^{\text{Ł}} p^{\Sigma} p^{\Pi} p^{\Omega}}{v_2^4} \right] ,
\end{aligned}
\tag{19}
$$

where all $\alpha^{(n)}$ are functions of ϕ^a.

Very schematically we present, as well, the second order corrections to the entropy function

$$
\begin{aligned}
&\mathcal{E}_{R^3} \sim v_1 v_2 \left[\frac{1}{v_1^3} + \frac{1}{v_1^2 v_2} + \frac{1}{v_1 v_2^2} + \frac{1}{v_2^3} \right] , \quad \mathcal{E}_{RF^4} \sim v_1 v_2 \left[\frac{1}{v_1} + \frac{1}{v_2} \right] \left[\frac{e^4}{v_1^4} + \frac{e^3 p}{v_1^3 v_2} + \ldots + \frac{p^4}{v_2^4} \right] , \\
&\mathcal{E}_{R^2F^2} \sim v_1 v_2 \left[\frac{1}{v_1^2} + \frac{1}{v_1 v_2} + \frac{1}{v_2^2} \right] \left[\frac{e^2}{v_1^2} + \frac{ep}{v_1 v_2} + \frac{p^2}{v_2^2} \right] , \quad \mathcal{E}_{F^6} \sim v_1 v_2 \left[\frac{e^6}{v_1^6} + \frac{e^5 p}{v_1^5 v_2} + \ldots + \frac{p^6}{v_2^6} \right] .
\end{aligned}
\tag{20}
$$

Here the coefficients are assumed to depend on the scalar fields ϕ^a, but their dependence is not written explicitly for the sake of simplicity. The function $\mathcal{E}_2$ (13), in turn, is given by the sum

$$
\mathcal{E}_2 = \mathcal{E}_{R^3} + \mathcal{E}_{R^2F^2} + \mathcal{E}_{RF^4} + \mathcal{E}_{F^6} .
$$

Each of the functions of eqs. (19) and (20) turns out to be an eigenfunction of the operator appearing in the expression for the effective potential (17)

$$
\left[v_1 \frac{\partial}{\partial v_1} + v_2 \frac{\partial}{\partial v_2} \right] \mathcal{E}_{R^n F^m} = (2 - n - m)\, \mathcal{E}_{R^n F^m} .
$$

This allows us to rewrite the effective potential (17) in the following form:

$$
\pi V_{eff}(\phi) = \mathcal{E}(e, v, \phi) - \frac{\alpha'^2}{4\pi} \left[\mu^{\text{Ł}\Sigma} \frac{\partial \mathcal{E}_1}{\partial e^{\text{Ł}}} \frac{\partial \mathcal{E}_1}{\partial e^{\Sigma}} + \frac{2}{V_{BH}} \left(\mathcal{E}_{RF^2} + 2\, \mathcal{E}_{F^4} \right) \left(v_1 \frac{\partial \mathcal{E}_1}{\partial v_1} + e^{\text{Ł}} \frac{\partial \mathcal{E}_1}{\partial e^{\text{Ł}}} \right) \right] , \tag{17$'$}
$$

where, as before, the right hand side is supposed to be taken on the solution (9).

3.2 Corrections to the value of the entropy

In the previous section we derived the effective black hole potential which encodes the information about the entropy of a black hole. Now we are going to calculate the value of the entropy up to the second order in α'. So, we should extremize the effective black hole potential with respect to the scalar fields

$$
\frac{\partial V_{eff}}{\partial \phi^a} = 0. \tag{21}
$$

In order to resolve these equations perturbatively let us expand the effective potential and the scalar fields in series on α' up to the second order

$$
V_{eff} = V_0 + \alpha' V_1 + \alpha'^2 V_2 + O(\alpha'^3), \qquad \phi^a = \phi^a_0 + \alpha' \phi^a_1 + \alpha'^2 \phi^a_2 + O(\alpha'^3). \tag{22}
$$

The expansion of V_{eff} is nothing but a concise form of the eq. (17′). Being written in full details it gives

$$
\begin{aligned}
&V_0 = \pi^{-1}\,\mathcal{E}_0 = V_{BH}, \qquad V_1 = \pi^{-1}\,\mathcal{E}_1, \\
&V_2 = \frac{1}{\pi}\,\mathcal{E}_2 - \frac{1}{4\pi^2}\left[\mu^{\text{L}\Sigma}\,\frac{\partial\mathcal{E}_1}{\partial e^{\text{L}}}\frac{\partial\mathcal{E}_1}{\partial e^{\Sigma}} + \frac{2}{V_{BH}}\,\left(\mathcal{E}_{RF^2} + 2\,\mathcal{E}_{F^4}\right)\left(v_1\frac{\partial\mathcal{E}_1}{\partial v_1} + e^{\text{L}}\frac{\partial\mathcal{E}_1}{\partial e^{\text{L}}}\right)\right]
\end{aligned}
\tag{23}
$$

where all $\mathcal{E}$-terms are calculated on the "classical" solution (9). After substitution of the expansions (22) into the criticality condition (21) one immediately derives

$$
\begin{aligned}
&\left.\frac{\partial V_0}{\partial\phi^a}\right|_0 = 0, && (a)\\
&\left.\frac{\partial^2 V_0}{\partial\phi^a\partial\phi^b}\right|_0 \phi_1^b = -\left.\frac{\partial V_1}{\partial\phi^a}\right|_0, && (b)\\
&\left.\frac{\partial^2 V_0}{\partial\phi^a\partial\phi^b}\right|_0 \phi_2^b = -\left.\frac{\partial V_2}{\partial\phi^a}\right|_0 - \left.\frac{\partial^2 V_1}{\partial\phi^a\partial\phi^b}\right|_0 \phi_1^b - \frac{1}{2}\left.\frac{\partial^3 V_0}{\partial\phi^a\partial\phi^b\partial\phi^c}\right|_0 \phi_1^b\phi_1^c, && (c)
\end{aligned}
\tag{24}
$$

where the subscript zero means that the corresponding expression should be taken upon $\phi^a = \phi_0^a$. Although, in order to obtain an extreme value of a function up to α'^n order, it is enough to know a solution to criticality condition up to $\alpha'^{(n-1)}$ order, we wrote down as well an equation (24c) defining the second order solution ϕ_2^a. It relates to a subtle effect that we illustrate on a simpler example.

Let us suppose that we are interested in calculating the value of the entropy up to α'^1 order. In this case we need to know only a "classical" solution ϕ_0^a to the eq. (24a), so that one might think that eqs. (24b) are of no importance, since they define the value of the first order solution ϕ_1^a. The subtlety comes out when the "classical" black hole potential V_0 has a flat direction. In this case the matrix of the second derivatives (which is nothing but a Hessian matrix) becomes degenerate and not all of ϕ_0^a might be determined from (24a). The undetermined scalar fields appear then in the right hand side of (24b). This fact might make the system of eqs. (24b) to become inconsistent, since the Hessian matrix is degenerate. Considering this possibility as quite unphysical, we impose the condition that (24b) be consistent. This condition might fix some of the previously undetermined ϕ_0^a. If the eqs. (24b) turns out to be consistent identically, it means that the symmetry of the first order corrections to the effective potential coincides with the symmetry of the "classical" effective potential.

When one is interested in corrections up to α'^2, then one has to check also that (24c) is consistent. The extreme value of the effective potential (that is, the entropy in fact) is then given by the expression

$$
V_{eff.extr} = V_0(\phi_0) + \alpha' V_1(\phi_0) + \alpha'^2\left[V_2(\phi_0) - \frac{1}{2}\left.\frac{\partial V_1}{\partial\phi^a}\right|_0 \phi_1^a\right]. \tag{25}
$$

Let us analyse this expression. First of all, the black hole potential (2) is a homogeneous function of degree two in the charges p^{L} and q_{L}. Then, the "classical" solution ϕ_0 is homogeneous of zero degree on the charges. It immediately follows from the fact that V_{BH} is homogeneous and ϕ_0 is a solution to a homogeneous equation (4). In order to write down the dependence of the entropy on the charges, let us for simplicity combine them to form a symplectic charge vector

$$
P^{\text{L}} = \left(p^{\text{L}}, q_{\text{L}}\right).
$$

Calculating the expressions $\mathcal{E}_{1,2}$ on the "classical" solutions according to formulae (19) and (20) and taking into account eqs. (23), (24) and (25), one gets

$$
\begin{aligned}
S = S_0 + \alpha' \left[A + \frac{1}{S_0} A_{\scriptscriptstyle \Lambda\Sigma} P^{\scriptscriptstyle \Lambda} P^{\scriptscriptstyle \Sigma} + \frac{1}{S_0^2} A_{\scriptscriptstyle \Lambda\Sigma\Pi\Omega} P^{\scriptscriptstyle \Lambda} P^{\scriptscriptstyle \Sigma} P^{\scriptscriptstyle \Pi} P^{\scriptscriptstyle \Omega} \right] + \frac{\alpha'^2}{S_0} \Big[B + \frac{1}{S_0} B_{\scriptscriptstyle \Lambda\Sigma} P^{\scriptscriptstyle \Lambda} P^{\scriptscriptstyle \Sigma} \\
+ \frac{1}{S_0^2} B_{\scriptscriptstyle \Lambda\Sigma\Pi\Omega} P^{\scriptscriptstyle \Lambda} P^{\scriptscriptstyle \Sigma} P^{\scriptscriptstyle \Pi} P^{\scriptscriptstyle \Omega} + \frac{1}{S_0^3} B_{\scriptscriptstyle \Lambda_1 \ldots \Lambda_6} P^{\scriptscriptstyle \Lambda_1} \ldots P^{\scriptscriptstyle \Lambda_6} \Big] + O(\alpha'^3).
\end{aligned}
\tag{26}
$$

The coefficients $A, A_{\scriptscriptstyle \Lambda\Sigma}, A_{\scriptscriptstyle \Lambda\Sigma\Pi\Omega}$ and $B, B_{\scriptscriptstyle \Lambda\Sigma}, \ldots$ are composed of the previously introduced functions $\alpha^{(n)}$, $\mu_{\scriptscriptstyle \Lambda\Sigma}$ and $\nu_{\scriptscriptstyle \Lambda\Sigma}$ and in general depend on the "classical" values of the scalar fields that are homogeneous of the zero degree on the charges. The explicit form of these coefficients depends on the model one considers and, if one considers higher dimensional theory, on the possible compactifications to four dimensions. Nevertheless, the structure of the contributions coming from the higher derivatives remains the same.

The generalization of this formula for higher order derivative corrections is straightforward. We see that really all the series is built out of two quantities: the classical value of the entropy (proportional to V_{BH}), which is quadratic in the charges for the extremal black holes in $D = 4$, and the classical values of the scalar fields, which are homogeneous of degree zero.

Formula (26) is an agreement with the result obtained for the one-loop correction to the Bekenstein–Hawking entropy [14] when only the R^2 term is present in the action ($1'$).

As we have noticed before, the coefficients A and B (with any number of indices) in (26) depend on the classical values of the scalar fields. If the black hole potential V_{BH} has a flat direction, this means that not all of the scalar fields get fixed. This is not a problem for the V_{BH}, since its value does not depend on the scalars which are not fixed. But the first order correction might depend on these not fixed values. This means that black hole entropy depends on the values of the non-fixed scalars which can change continuously. This fact threatens the attractor mechanism paradigm and can induce a violation of the second law of black hole thermodynamics. Really, such a violation does not occur and the attractor mechanism paradigm is preserved. As it has been already explained, once a flat direction is present, the system of eqs. (24b) becomes degenerate and to possess a solution its right hand side should satisfy some consistency condition, which might fix some of the previously free scalar fields ϕ_0^a. If not all of the scalars ϕ_0^a get fixed, then the group symmetry of the higher derivative corrections is a subgroup of the symmetry of the black hole potential, hence the higher order corrections will not depend on these non-fixed scalars.

Looking at the eq. (26) one sees that higher order corrections become singular when considering small black holes. Once V_{BH} is equal to zero, the perturbative expansion is not valid anymore. And there arises a question whether quantum effects might cure small black holes. This will the subject of a forthcoming analysis, whereas for the time being we limit ourselves to just some remarks. In realistic models the singularity might be removed when along with $S_0 = 0$ the coefficients A and B tend to zero as $\phi^a = \phi_0^a$ in a way such that the corresponding fractions in (26) remain finite.

4 Application to the *stu* model

In order to shed light on specific features of the effect of quantum corrections in the case of small black holes or when flat directions are present, we are going to revisit some well known examples where both such features are present. In particular we consider the *stu* model

with higher order corrections stemming from compactification of the heterotic string theory to four dimensions [7, 10]. The classical non-BPS solution for this model possesses two flat directions [15] in the axionic sector of its moduli space. In addition, this example allows us to investigate small black holes.

In order to derive the form of the higher order corrections to this model we take the action of the heterotic string model [13], with all α' order terms calculated, and truncate it to six dimensions. Being afterwards reduced to four dimensions it corresponds to the *stu* model with higher order derivative terms.

We perform our computations directly in six dimensions since dimensional reduction of the α' corrections to four dimensions might be a topic of independent research. Moreover, working in six dimensions makes it easier to take into consideration the gravitational Chern–Simons term [7, 10].

We will follow the strategy given in [10] and, in order to simplify the comparison of the results, we will mostly follow the notations by [10]. The difference of our consideration from that of [10] is that we preserve all the scalar fields – axions and dilatons – present in the theory, since the axion fields turn out to play an important role when it comes to the question of stability.

The action [13] we start with is the bosonic part of the heterotic string theory action truncated to six dimensions. It describes the coupling of a two-form field B_{MN} and a dilaton field Φ to six-dimensional Einstein gravity

$$S = \frac{1}{32\pi}\int d^6x\sqrt{-G}\,e^{-\Phi}\left[R + \partial_M\Phi\partial^M\Phi - \frac{1}{12}H_{MNK}H^{MNK} + \frac{\alpha'}{8}\,\bar{R}_{MNKL}\bar{R}^{MNKL} + \alpha'^3\Delta\mathcal{L}_3\right] + \dots \tag{27}$$

where

$$\bar{R}^M{}_{NPQ} = R^M{}_{NPQ} + \nabla_{[P}H^M{}_{Q]N} - \frac{1}{2}H^M{}_{R[P}H^R{}_{Q]N}\,. \tag{28}$$

Here the term $\Delta\mathcal{L}_3$ is of the forth order in the curvature $\bar{R}_{MNKL}$. The field strength H_{PQR} includes also a gravitational Chern–Simons term

$$\begin{aligned}
H_{PQR} &= \partial_P B_{QR} + \partial_Q B_{RP} + \partial_R B_{PQ} - 3\alpha'\left(\Omega_{PQR} + \mathcal{A}_{PQR}\right),\\
\Omega_{PQR} &= \frac{1}{2}\,\Gamma^N_{PM}\partial_Q\Gamma^M_{NR} + \frac{1}{3}\,\Gamma^N_{PM}\Gamma^M_{QK}\Gamma^K_{NR} + \text{antisym. on } P, Q, R,\\
\mathcal{A}_{PQR} &= \frac{1}{4}\,\partial_P\left(\Gamma^M_{QN}H^N{}_{MR}\right) + \frac{1}{8}\,H^M{}_{PN}\nabla_Q H^N{}_{MR} - \frac{1}{4}\,R^{MN}{}_{PQ}H^{RMN}\\
&\quad + \frac{1}{24}\,H^M{}_{PN}H^S{}_{QM}H^N{}_{RS} + \text{ antisym. on } P, Q, R.
\end{aligned} \tag{29}$$

As prescribed in [10], one may dualize the field strength H_{PQR} into a new one K_{PQR} which is an exact three-form

$$K_{PQR} = \partial_P C_{QR} + \partial_Q C_{RP} + \partial_R C_{PQ},$$

obeying, obviously, the Bianchi identities. To this end, one adds the term

$$\int d^6x\varepsilon^{MNKPQR}K_{MNK}\left(H_{PQR} + 3\alpha'\Omega_{PQR}\right)$$

into the original action (27) and eliminates the field strength H_{PQR}. We postpone the elimination of H_{PQR} for a while, for a reason to be clarified later.

Dealing with the Chern-Simons term is a little bit tricky [10] and consists in maintaining the Lorentz covariance of the Lagrangian. We succeeded in singling out a manifestly covariant part of the Lagrangian by adding total derivative terms, in such a way that the second derivatives of the four dimensional electromagnetic potentials originating in the six dimensional metric disappear.

In order to be able to interpret the model (27) as an ancestor of the stu model, one should clarify the field content of the theory. First of all, the six dimensional metric tensor[1] G_{MN} when reduced to four dimensions produces a four dimensional metric tensor $g_{\mu\nu}$, two vector fields $A_\mu^{1,2}$ and three scalar fields – two dilatons u_1, u_2 and one axion c. Then, the two-form potential C_{MN} produces a four dimensional two-form $C_{\mu\nu}^{(4)}$, two vector fields $A_\mu^{3,4}$ and one axion $C_{mn} = b\,\epsilon_{mn}$. In four dimensions a two-form $C_{\mu\nu}^{(4)}$ is dual to a scalar; this duality gives rise to another axion a. Finally, the original scalar field Φ gives rise to a four dimensional dilaton u_s. So, we end up with a four dimensional metric tensor, four vector fields and six real scalars – three axions a, b, c and three dilatons u_1, u_2, u_s (more precisely, exponentials of the dilatons).

The entropy function formalism in this case is based on putting the theory on a six-dimensional background

$$ds_{(6)}^2 = ds_{(4)}^2 + G_{mn}\left(dx^m + A_\mu^{m-3}dx^\mu\right)\left(dx^n + A_\nu^{n-3}dx^\nu\right), \tag{30}$$

where the two-dimensional metric tensor G_{mn} is parameterized as follows [16]:

$$G_{mn} = \begin{pmatrix} u_1^2 + c^2\,u_2^2 & -c\,u_2^2 \\ -c\,u_2^2 & u_2^2 \end{pmatrix}. \tag{31}$$

The connections A_μ^{m-3} are chosen in the following form:

$$A_\mu^1 = \left(2\,r\,e^1, 0, 0, 0\right), \qquad A_\mu^2 = (2\,r\,e^2, 0, 0, -\frac{p^2}{2\pi}\cos\theta). \tag{32}$$

Hence, in what follows they will give the so-called magnetic configuration corresponding to the $D0 - D4$ branes. The other two vector fields coming from C_{MN} are equal to

$$A_\mu^3 = \left(\frac{1}{8}\,r\,e^3, 0, 0, 0\right), \qquad A_\mu^4 = \left(\frac{1}{8}\,r\,e^4, 0, 0, -\frac{p^4}{32\pi}\cos\theta\right). \tag{33}$$

The electric potentials e^{L} are dynamical fields; therefore we do not put them equal to zero ad hoc, as their values are to be fixed when minimizing the entropy function.

In the entropy function formalism one should fix the values of the scalars at the horizon of the black hole. Thus we parameterize the dilaton field as follows [10]:

$$e^{-\Phi} = \frac{u_s}{64\pi^2 u_1 u_2}$$

From a four dimensional perspective the field strength K_{MNK} is related to the vector potentials and axions in the following way:

$$\begin{aligned} K_{\mu\nu\rho} &= \frac{u_s}{64\pi^2 u_1^2 u_2^2}\sqrt{-g}g^{\sigma\tau}\epsilon_{\tau\mu\nu\rho}\,a_{,\sigma} + 3K_{[\mu\nu n}A_{\lambda]}^{n-3} + 3K_{[\mu mn}A_\nu^{m-3}A_{\lambda]}^{n-3} \\ K_{\mu\nu m} &= -F_{\mu\nu}^{m-1} - b\,\epsilon_{mn}F_{\mu\nu}^{n-3} + 2A_{[\mu}^{n-3}K_{\nu]mn} \\ K_{\mu mn} &= b_{,\mu}\,\epsilon_{mn} \end{aligned}$$

[1] we assume the splitting of the six dimensional indices $M, N, K, \ldots = 0, 1, \ldots, 5$ into four dimensional $\mu, \nu, \rho, \ldots = 0, 1, 2, 3$ and two dimensional $m, n, k, \ldots = 4, 5$ ones

and, therefore, its non-zero components are given by the expressions

$$\begin{aligned} K_{013} &= \frac{1}{16\pi}\, p^2(16b\, e^1 - e^4)\cos\theta, & K_{014} &= \frac{e^3}{8} + 2b\, e^2, & K_{015} &= \frac{e^4}{8} - 2b\, e^1, \\ K_{023} &= -\frac{1}{16\pi}\,(16b\, e^1 p^2 + e^4 p^4) r\sin\theta, & K_{234} &= -\frac{bp^2}{2\pi}\sin\theta, & K_{235} &= -\frac{p^4}{32\pi}\sin\theta. \end{aligned} \tag{34}$$

As it has been mentioned above, the field strength $H_{_{MNK}}$ should be eliminated by means of its equations of motion. Since we perform our calculations up to α'^2 order, one has to resolve these equations of motion up to the α'^2 order, what seems to be quite sophisticated due to the nonlinear nature of the action (27). In order to eliminate $H_{_{MNK}}$, we perform the following trick [7]: we use the Ansatz (dictated by the symmetry of the problem) for its non-vanishing components

$$\begin{aligned} H_{013} &= h_1\cos\theta, & H_{014} &= h_2, & H_{015} &= -\frac{2\pi}{p^2} h_1, \\ H_{023} &= h_3 r\sin\theta, & H_{234} &= h_4\sin\theta, & H_{235} &= h_5\sin\theta \end{aligned} \tag{35}$$

and include the fields h_i in the set of dynamical fields, with respect to which the entropy function is to be minimised

$$\begin{aligned} &\frac{\partial\mathcal{E}}{\partial h_i} = \frac{\partial\mathcal{E}}{\partial e^{\mathrm{L}}} = \frac{\partial\mathcal{E}}{\partial v_1} = \frac{\partial\mathcal{E}}{\partial v_2} = 0, & (a) \\ &\frac{\partial\mathcal{E}}{\partial a} = \frac{\partial\mathcal{E}}{\partial b} = \frac{\partial\mathcal{E}}{\partial c} = \frac{\partial\mathcal{E}}{\partial u_1} = \frac{\partial\mathcal{E}}{\partial u_2} = \frac{\partial\mathcal{E}}{\partial u_s} = 0. & (b) \end{aligned} \tag{36}$$

The entropy function acquires the form

$$\begin{aligned} \mathcal{E} = 2\pi \Bigg(& e^1 q_1 + e^3 q_3 + \frac{u_s v_2}{4} - \frac{u_s v_1}{4} - 16bh_1\pi - 32bh_4\pi e^1 - 32bh_5\pi e^2 - 2h_5\pi e^3 \\ & + 2h_4\pi e^4 \frac{ae^4p^2}{4\pi} - \frac{h_2p^4}{2} + \frac{ae^2p^4}{4\pi} - \frac{h_3^2u_s}{16v_2} + \frac{h_3h_4e^1u_s}{4v_2} - \frac{h_4^2(e^1)^2u_s}{4v_2} + \frac{h_3h_5e^2u_s}{4v_2} \\ & - \frac{h_4h_5e^1e^2u_s}{2v_2} - \frac{h_5^2(e^2)^2u_s}{4v_2} + \frac{h_4^2u_sv_1}{16u_1^2v_2} + \frac{ch_4h_5u_sv_1}{8u_1^2v_2} + \frac{h_5^2u_sv_1}{16u_2^2v_2} + \frac{c^2h_5^2u_sv_1}{16u_1^2v_2} \\ & + \frac{u_2^2(p^2)^2u_sv_1}{64\pi^2v_2} - \frac{h_2^2u_sv_2}{16u_1^2v_1} - \frac{u_1^2e_1^2u_sv_2}{4v_1} - \frac{c^2u_2^2e_1^2u_sv_2}{4v_1} + \frac{cu_2^2e_1e_2u_sv_2}{2v_1} \\ & - \frac{u_2^2e_2^2u_sv_2}{4v_1} - \frac{c^2\pi^2h_1^2u_sv_2}{4u_1^2(p^2)^2v_1} - \frac{\pi^2h_1^2u_sv_2}{4u_2^2(p^2)^2v_1} + \frac{c\pi h_1h_2u_sv_2}{4u_1^2p^2v_1} \Bigg) + O(\alpha') \,. \end{aligned} \tag{37}$$

Here, for the sake of brevity, we avoid reporting the explicit expression we derived for the first α' order corrections. When minimizing this expression, we perform an expansion of all dynamical fields over α', i.e.

$$a = a_{(0)} + \alpha' a_{(1)} + \ldots, \qquad b = b_{(0)} + \alpha' b_{(1)} + \ldots, \qquad \text{etc.}$$

In this way one gets “classical” solutions to (36)

$$
\begin{aligned}
e^1_{(0)} &= \frac{1}{4\pi q_1}\sqrt{-p^2p^4q_1q_3}, \quad e^2_{(0)} = e^4_{(0)} = 0, \quad e^3_{(0)} = \frac{1}{4\pi q_3}\sqrt{-p^2p^4q_1q_3},\\
v_1^{(0)} &= v_2^{(0)} = \frac{1}{8\pi^2}\left(e^{-\alpha_2}+e^{\alpha_2}\right)p^2q_3\\
h_1^{(0)} &= -h_3^{(0)} = -\frac{1-e^{2\alpha_1}}{1+e^{2\alpha_1}}\frac{p^2q_3}{4\pi^2}, \quad h_2^{(0)} = -\frac{1}{2\pi p^4}\sqrt{-p^2p^4q_1q_3},\\
h_4^{(0)} &= -\frac{1-e^{2\alpha_1}}{1+e^{2\alpha_1}}\frac{\sqrt{-p^2p^4q_1q_3}}{2\pi p^4}, \qquad h_5^{(0)} = \frac{q_3}{2\pi},\\
a_{(0)} &= 4\pi\frac{1-e^{2\alpha_1}}{1+e^{2\alpha_1}}\sqrt{-\frac{q_1q_3}{p^2p^4}}, \quad b_{(0)} = -\frac{1}{16}\frac{1-e^{2\alpha_2}}{1+e^{2\alpha_2}}\sqrt{-\frac{p^4q_1}{p^2q_3}},\\
c_{(0)} &= -\frac{1-e^{2\alpha_3}}{1+e^{2\alpha_3}}\sqrt{-\frac{p^2q_1}{p^4q_3}}, \quad u_1^{(0)} = \sqrt{\frac{4\,e^{\alpha_1+\alpha_3}}{(1+e^{2\alpha_1})(1+e^{2\alpha_3})}}\sqrt{-\frac{q_1}{p^4}},\\
u_2^{(0)} &= \sqrt{\frac{e^{\alpha_1}(1+e^{2\alpha_3})}{e^{\alpha_3}(1+e^{2\alpha_1})}}\sqrt{\frac{q_3}{p^2}}, \quad u_s^{(0)} = \frac{16\pi e^{\alpha_2}}{1+e^{2\alpha_2}}\sqrt{-\frac{p^4q_1}{p^2q_3}}.
\end{aligned}
\tag{38}
$$

Here the three real parameters α_i are restricted by one condition [17]

$$\alpha_1+\alpha_2+\alpha_3 = 0,$$

so that the presence of two independent unconstrained parameters indicates the presence of two flat directions.

The charges above are connected with standard stu black hole charges by the following transformations:

$$q_1 = 4\,\pi\,Q_0, \quad q_3 = -\frac{P^1}{2}, \quad p^2 = \frac{P^3}{2}, \quad p^4 = -4\,\pi\,P^2.$$

One can check this by making a transformation from $SO(2,2)$ to $(SU(1,1)/U(1))^3$ basis [17, 18].

Naïve substitution of the zero order solution into the entropy function makes it dependent on these parameters. This effect appears to contradict the attractor mechanism paradigm. However, a subtler analysis shows that these parameters get fixed and the correctness of the attractor mechanism is restored on the quantum level too.

To illustrate how the dependence on these parameters drops out, we present in details the first order correction to the entropy. So, the “classical” solution (38) yields the value of the entropy up to the α'^1 order

$$\mathcal{E} = \sqrt{-p^2p^4q_1q_3} + 16\pi^2\alpha' e^{\alpha_2}\frac{3+2e^{2\alpha_2}+3e^{4\alpha_2}}{(1+e^{2\alpha_2})^3}\sqrt{-\frac{p^4q_1}{p^2q_3}} + O(\alpha'^2).$$

Let us note that even including axions the non-BPS solution the first order correction to the entropy does not contain any contribution form the Chern–Simons term, in an agreement with [10]. Sticking to the first order approximation for the entropy, the only thing to know about the first order solution to the eq. (36) is that it exists. In general, this is not guaranteed automatically, since the matrix of the second derivatives of the entropy function is degenerate. In the case under consideration the first order solution exists if

$$\alpha_2 = 0.$$

In this case the first order solution acquires the following form:

$$h_1^{(1)} = \frac{a_1 p^2\sqrt{-p^2p^4q_1q_3}}{16\pi^3 q_1}, \quad h_2^{(1)} = -\frac{32e^{2\alpha_1}\pi q_1}{(1+e^{2\alpha_1})^2\sqrt{-p^2p^4q_1q_3}}, \quad h_5^{(1)} = -\frac{8\left(-1+e^{2\alpha_1}\right)^2\pi}{\left(1+e^{2\alpha_1}\right)^2 p^2},$$

$$h_3^{(1)} = \frac{\left(1+e^{2\alpha_1}\right)a_1(p^2)^2p^4q_3 - 96\left(-1+e^{2\alpha_1}\right)\pi^3\sqrt{-p^2p^4q_1q_3}}{16\left(1+e^{2\alpha_1}\right)\pi^3\sqrt{-p^2p^4q_1q_3}},$$

$$h_4^{(1)} = -\frac{\left(1+e^{2\alpha_1}\right)a_1(p^2)^2p^4q_3 + 64\left(-1+e^{2\alpha_1}\right)\pi^3\sqrt{-p^2p^4q_1q_3}}{8\left(1+e^{2\alpha_1}\right)\pi^2p^2p^4q_3},$$

$$e^1_{(1)} = -\frac{4\pi p^4}{\sqrt{-p^2p^4q_1q_3}}, \qquad e^3_{(1)} = \frac{4\pi p^4 q_1}{q_3\sqrt{-p^2p^4q_1q_3}}, \qquad e^2_{(1)} = e^4_{(1)} = 0, \quad v_1^{(1)} = v_2^{(1)} = 0,$$

$$u_1^{(1)} = \frac{e^{-\alpha_1}}{8\left(1+e^{2\alpha_1}\right)^3\pi p^2(p^4)^2\sqrt{-\frac{q_1}{p^4}}q_3}\Big[-128e^{2\alpha_1}\left(-1+e^{4\alpha_1}\right)\pi b_1p^2q_3\sqrt{-p^2p^4q_1q_3}+$$
$$+p^4\left(128e^{2\alpha_1}\left(1+6e^{2\alpha_1}+e^{4\alpha_1}\right)\pi^3q_1+\left(-1+e^{2\alpha_1}\right)\left(1+e^{2\alpha_1}\right)^3a_1p^2\sqrt{-p^2p^4q_1q_3}\right)\Big],$$

$$u_2^{(1)} = \frac{8\left(-1+e^{2\alpha_1}\right)\left(\left(1+e^{2\alpha_1}\right)b_1(p^2)^2q_3^2 - \left(-1+e^{2\alpha_1}\right)\pi^2\sqrt{-p^2p^4q_1q_3}\right)}{\left(1+e^{2\alpha_1}\right)^2(p^2)^2\sqrt{\frac{q_3}{p^2}}\sqrt{-p^2p^4q_1q_3}},$$

$$c_{(1)} = -\frac{p^2q_1}{16\left(1+e^{2\alpha_1}\right)^3\pi\left(-p^2p^4q_1q_3\right)^{3/2}}\Big[-1024e^{2\alpha_1}\left(1+e^{2\alpha_1}\right)\pi b_1p^2q_3\sqrt{-p^2p^4q_1q_3}+$$
$$+p^4\left(-1024e^{2\alpha_1}\left(-1+e^{2\alpha_1}\right)\pi^3q_1+4\left(1+e^{2\alpha_1}\right)^3a_1p^2\sqrt{-p^2p^4q_1q_3}\right)\Big],$$

$$u_s^{(1)} = -\frac{128\pi^3\sqrt{-p^2p^4q_1q_3}}{(p^2)^2q_3^2},$$

Let us note that the two scalar fields (in our case these are $a_{(1)}$ and $b_{(1)}$) remain undefined in this approximation. With vanishing α_2 the first order correction to the entropy does not depend on any free parameter anymore and the second order correction acquires a much simpler form, so that we may write it down as

$$\mathcal{E} = \sqrt{-p^2p^4q_1q_3} + 16\pi^2\alpha'\sqrt{-\frac{p^4q_1}{p^2q_3}} - 16\pi^4\alpha'^2\frac{p^4q_1}{p^2q_3}\frac{1-34e^{\alpha_1}+e^{4\alpha_1}}{\sqrt{-p^2p^4q_1q_3}\left(1+e^{2\alpha_1}\right)^2} + O(\alpha'^3).$$

Performing analogous steps to second order, one can check that requiring the existence of the second order corrections to the solution of eqs. (36) yields $\alpha_1 = 0$. The solution we obtained in this way, corresponding to $\alpha_i = 0$, coincides with the axion free solution given in [7, 10]. The only difference is that we derived this solution keeping the full axion dynamics, without truncation. It is only on the horizon that the axion contribution vanishes.

The value of the entropy up to the second order is then given by

$$\mathcal{E} = \sqrt{-p^2p^4q_1q_3} + 16\pi^2\alpha'\sqrt{-\frac{p^4q_1}{p^2q_3}} - 128\pi^4\alpha'^2\frac{p^4q_1}{p^2q_3}\frac{1}{\sqrt{-p^2p^4q_1q_3}} + O(\alpha'^3). \qquad (39)$$

One can see that this expression shares the general features pointed out in (26).

Exact solutions

Despite the cumbersome and entangled structure of the above non-BPS perturbative solutions to the extremization condition of the entropy function, one can cast them in exact form[2]

$$
\begin{aligned}
&v_1 = v_2 = \frac{p^2 q_3}{4\pi^2}, \\
&h_2 = -\frac{\sqrt{-p^4 q_1 p^2 q_3}}{2\pi p^4}\,\frac{1}{\sqrt{1+\frac{32\pi^2\alpha'}{p^2 q_3}}}, && h_5 = \frac{q_3}{2\pi}, \\
&e^1 = \frac{\sqrt{-p^4 q_1 p^2 q_3}}{4\pi q_1}\sqrt{1+\frac{32\pi^2\alpha'}{p^2 q_3}}, && e^3 = \frac{\sqrt{-p^4 q_1 p^2 q_3}}{4\pi q_3}\,\frac{1}{\sqrt{1+\frac{32\pi^2\alpha'}{p^2 q_3}}}, \\
&u_1 = \sqrt{-\frac{q_1}{p^4}}\,\frac{1}{\sqrt{1+\frac{32\pi^2\alpha'}{p^2 q_3}}}, \quad u_2 = \sqrt{\frac{q_3}{p^2}}, && u_s = -\frac{8\pi p^4 q_1}{\sqrt{-p^4 q_1 p^2 q_3}}\,\frac{1}{\sqrt{1+\frac{32\pi^2\alpha'}{p^2 q_3}}},
\end{aligned}
\tag{40}
$$

for the charges

$$q_1 < 0, \quad q_3 > 0, \quad p^2 > 0, \quad p^4 > 0. \tag{41}$$

Once the close form of the solution is obtained, one can see that the perturbation expansion that we were performing is valid when

$$\frac{32\,\pi^2\alpha\,'}{|p^2 q_3|} \ll 1.$$

If one fixes the value of $\alpha\,'$, then one may understand this formula as a condition on the charges

$$|p^2 q_3| \gg 1, \tag{42}$$

when classical effects become dominant. As we will see later, the condition (42) fails for small black holes.

In the paper [7] another exact non-BPS solution was found, which in our notations reads[2]

$$
\begin{aligned}
&v_1 = v_2 = -\frac{p^2 q_3}{4\pi^2}\left(1-\frac{32\pi^2\alpha\,'}{p^2 q_3}\right), \\
&h_2 = -\frac{\sqrt{-p^4 q_1 p^2 q_3}}{2\pi p^4}\sqrt{1-\frac{32\pi^2\alpha'}{p^2 q_3}}, && h_5 = \frac{q_3}{2\pi}\left(1-\frac{32\pi^2\alpha\,'}{p^2 q_3}\right), \\
&e^1 = \frac{\sqrt{-p^4 q_1 p^2 q_3}}{4\pi q_1}\sqrt{1-\frac{32\pi^2\alpha'}{p^2 q_3}}, && e^3 = \frac{\sqrt{-p^4 q_1 p^2 q_3}}{4\pi q_3}\,\frac{1}{\sqrt{1-\frac{32\pi^2\alpha'}{p^2 q_3}}}, \\
&u_1 = \sqrt{\frac{q_1}{p^4}}, \quad u_2 = \sqrt{-\frac{q_3}{p^2}}\sqrt{1-\frac{32\pi^2\alpha'}{p^2 q_3}}, && u_s = \frac{8\pi p^4 q_1}{\sqrt{-p^4 q_1 p^2 q_3}}\,\frac{1}{\sqrt{1-\frac{32\pi^2\alpha'}{p^2 q_3}}},
\end{aligned}
\tag{43}
$$

which is valid for the following charges:

$$q_1 < 0, \quad q_3 < 0, \quad p^2 > 0, \quad p^4 < 0. \tag{44}$$

[2]only non-vanishing fields are written down

Both solutions correspond to the same value of the entropy

$$S = \sqrt{-p^2p^4q_1q_3}\sqrt{1 + \frac{32\pi^2\alpha'}{|p^2q_3|}}. \tag{45}$$

In the classical limit $\alpha' = 0$ the Hessian matrix of the solutions (40) and (43) has the following eigenvalues:

$$\begin{aligned} &\lambda_1 = \lambda_2 = 0, \quad \lambda_3 = 2|p^4|\sqrt{-\frac{p^2p^4q_3}{q_1}}, \quad \lambda_4 = \frac{|p^2q_3|}{64\pi^2}\sqrt{-\frac{p^2q_3}{p^4q_1}}, \\ &\lambda_5 = 2p^2\sqrt{-\frac{p^2p^4q_1}{q_3}}, \quad \lambda_6 = \frac{(p^4)^2q_3^2 + 256(p^2)^2q_3^2 + \frac{(p^4)^2(p^2)^2}{16\pi^2}}{\sqrt{-p^2p^4q_1q_3}}. \end{aligned} \tag{46}$$

When turning on the quantum effects, one finds that the eigenvalue λ_1 remains equal to zero, while λ_2 acquires a positive correction

$$\lambda_2 = 32\pi^4|p^4|\sqrt{-\frac{p^4}{p^2q_1q_3}}\frac{(p^4)^2 + 4096\pi^2q_3^2}{16\pi^2(p^4)^2q_3^2 + (p^2)^2(p^4)^2 + 4096\pi^2q_3^2(p^2)^2}\,\alpha'^2 + O(\alpha'^3), \tag{47}$$

so that one can say that the solutions are "stable".

To complete the list of solutions, we present as well two additional solutions. One of them, found in [7], is a BPS solution (see footnote 2 on p.15)

$$\begin{aligned} &v_1 = v_2 = -\frac{p^2q_3}{4\pi^2}\left(1 - \frac{32\pi^2\alpha'}{p^2q_3}\right), \\ &h_2 = -\frac{\sqrt{p^2p^4q_1q_3}}{2\pi p^4}\frac{1 - \frac{32\pi^2\alpha'}{p^2q_3}}{\sqrt{1 - \frac{64\pi^2\alpha'}{p^2q_3}}}, \qquad h_5 = \frac{q_3}{2\pi}\left(1 - \frac{32\pi^2\alpha'}{p^2q_3}\right), \\ &e^1 = \frac{\sqrt{p^2p^4q_1q_3}}{4\pi q_1}\sqrt{1 - \frac{64\pi^2\alpha'}{p^2q_3}}, \qquad e^3 = \frac{\sqrt{p^2p^4q_1q_3}}{4\pi q_3}\frac{1}{\sqrt{1 - \frac{64\pi^2\alpha'}{p^2q_3}}}, \\ &u_1 = \sqrt{-\frac{q_1}{p^4}\frac{1 - \frac{32\pi^2\alpha'}{p^2q_3}}{1 - \frac{64\pi^2\alpha'}{p^2q_3}}}, \qquad u_2 = \sqrt{-\frac{q_3}{p^2}}\sqrt{1 - \frac{32\pi^2\alpha'}{p^2q_3}}, \quad u_s = \frac{8\pi\sqrt{\frac{p^4q_1}{p^2q_3}}}{\sqrt{1 - \frac{64\pi^2\alpha'}{p^2q_3}}} \end{aligned} \tag{48}$$

and is valid for the charges

$$q_1 > 0, \quad q_3 < 0, \quad p^2 > 0, \quad p^4 < 0.$$

The entropy is then given by

$$S = \sqrt{p^2p^4q_1q_3}\sqrt{1 + \frac{64\pi^2\alpha'}{|p^2q_3|}}. \tag{49}$$

The other solution acquires no α' corrections

$$\begin{aligned} &v_1 = v_2 = \frac{p^2q_3}{4\pi^2}, \quad e^1 = \frac{p^4}{4\pi}\sqrt{\frac{p^2q_3}{p^4q_1}}, \quad e^3 = \frac{p^2}{4\pi}\sqrt{\frac{p^4q_1}{p^2q_3}}, \\ &h_2 = -\frac{q_1}{2\pi}\sqrt{\frac{p^2q_3}{p^4q_1}}, \quad h_5 = \frac{q_3}{2\pi}, \quad u_1 = \sqrt{\frac{q_1}{p^4}}, \quad u_2 = \sqrt{\frac{q_3}{p^2}}, \quad u_s = 8\pi\sqrt{\frac{p^4q_1}{p^2q_3}}. \end{aligned} \tag{50}$$

It is valid for the charges

$$q_1 > 0, \quad q_3 > 0, \quad p^2 > 0, \quad p^4 > 0$$

and the corresponding entropy is equal

$$S = \sqrt{p^4 q_1 p^2 q_3}.$$

Despite the fact that it has no quantum corrections, this solution is non-BPS but with vanishing central charge. These two solutions are stable in the classical limit, and we suppose that quantum effects do not spoil this feature.

Small black holes

Out of the solutions derived in closed form in previous section, only two admit small black hole limits, i.e. a 1/2-BPS (48) and a non-BPS $Z \neq 0$ (43), solutions. The remaining two solutions are non-BPS and do not contain small black holes (one of them, i.e. the non-BPS $Z = 0$ solution, remains even unaffected by quantum corrections). By definition, a small black hole has vanishing classical entropy. In the solutions reported in the previous section, the small black hole limit corresponds to

$$q_3 = 0. \tag{51}$$

When this limit is considered, only the two solutions (48) and (43) remain regular, while for the remaining two the radii of the AdS_2 and S^2 spaces go to zero. Furthermore, the non-BPS $Z = 0$ solution (50) has vanishing entropy, whereas the non-BPS $Z \neq 0$ solution (40) has the same (non vanishing) value of the entropy as the solution (43).

One can easily understand that the perturbation theory over the parameter α' fails for small black holes. Indeed the genuine parameter to make a perturbative expansion is q_3 for small black holes and $1/q_3$ for large ones.

Taking the limit (51) in the solutions (43) one gets the non-BPS small black hole solution

$$\begin{aligned} &v_1 = v_2 = 8\alpha', \quad h_2 = 2\sqrt{2\alpha' \frac{q_1}{p^4}}, \quad h_5 = -\frac{16\pi\alpha'}{p^2}, \quad e^1 = -\sqrt{2\alpha' \frac{p^4}{q_1}}, \\ &e^3 = -\frac{p^2}{16\pi^2}\sqrt{\frac{p^4 q_1}{2\alpha'}}, \quad u_1 = \sqrt{\frac{q_1}{p^4}}, \quad u_2 = \frac{4\sqrt{2}\pi\sqrt{\alpha'}}{p^2}, \quad u_s = \sqrt{\frac{2p^4 q_1}{\alpha'}}, \end{aligned} \tag{52}$$

with the corresponding entropy

$$S = 4\pi\sqrt{2\,\alpha' p^4 q_1}.$$

Calculating the Hessian matrix on the solution (52), one obtains the following eigenvalues:

$$\begin{aligned} &\lambda_1 = 0, \quad \lambda_2 = -\frac{(p^2)^4(p^4)^2 + 36(8\pi)^6(p^2)^2\,(\alpha')^2 + 9(4\pi)^6(p^4)^2\,(\alpha')^2}{512\sqrt{2}\pi^3(p^2)^2\sqrt{p^4 q_1 \alpha'}}, \\ &\lambda_3 = -\frac{16\pi}{3}p^4\sqrt{2\alpha' \frac{p^4}{q_1}}, \quad \lambda_4 = -\frac{\sqrt{2}}{\pi}\frac{(p^2)^2 p^4 q_1 + 4\pi^2\alpha'^2}{\sqrt{p^4 q_1 \alpha'}}, \quad \lambda_5 = 0, \quad \lambda_6 = 0. \end{aligned} \tag{53}$$

One sees that $\lambda_{2,4} < 0$ and $\lambda_3 > 0$, which means that the solution (52) is not stable.

Now, taking the limit (51) in the solutions (48) one gets the BPS small black hole solution

$$
\begin{gathered}
v_1 = v_2 = 8\alpha', \quad h_2 = 2\sqrt{-\frac{q_1}{\alpha' p^4}}, \quad h_5 = -\frac{16\pi\alpha'}{p^2}, \quad e^1 = 2\sqrt{-\alpha'\frac{p^4}{q_1}}, \\
e^3 = -\frac{p^2}{32\pi^2}\sqrt{\frac{-p^4 q_1}{\alpha'}}, \quad u_1 = \sqrt{-\frac{q_1}{2p^4}}, \quad u_2 = \frac{4\sqrt{2}\pi\sqrt{\alpha'}}{p^2}, \quad u_s = \sqrt{-\frac{p^4 q_1}{\alpha'}}\,,
\end{gathered}
\tag{54}
$$

with the corresponding entropy

$$S = 8\pi\sqrt{-\alpha' p^4 q_1}.$$

In this case two of the eigenvalues become zero, three of them become positive and one becomes negative

$$0, \quad 0, \quad \frac{a_1 + \sqrt{a_1^2 + b_1}}{c_1}, \quad \frac{a_1 - \sqrt{a_1^2 + b_1}}{c_1}, \quad \frac{a_2 + \sqrt{a_2^2 + b_2}}{c_2}, \quad \frac{a_2 - \sqrt{a_2^2 + b_2}}{c_2},$$

where

$$
\begin{aligned}
a_1 &= 25((p^2)^2 p^4)^2 + 1145(4\pi)^6(256(p^2)^2 + (p^4)^2)\alpha'^2, & a_2 &= 7(p^2)^2 p^4 q_1 + 288\pi^2 (p^4)^2 \alpha' - 56\pi^2\alpha'^2, \\
b_1 &= -1294(8\pi)^6((p^2)^2 p^4)^2(256(p^2)^2 + (p^4)^2)\alpha'^2, & b_2 &= 12160\pi^2 (p^4)^2\left(-(p^2)^2 p^4 q_1 + 8\pi^2\alpha'^2\right)\alpha', \\
c_1 &= 2588(2\pi)^3(p^2)^2\sqrt{-p^4 q_1 \alpha'}, & c_2 &= 38\pi\sqrt{-p^4 q_1 \alpha'}\,.
\end{aligned}
$$

We can then conclude that, amazingly enough, for small black holes, all found solutions – BPS and non-BPS – are not stable.

5 Conclusion

In this paper we considered a correction to the black hole entropy due to the most general higher order derivative terms in the Einstein–Maxwell action. We demonstrated that the general form of the corrections to the entropy is in agreement with previously found results [14]. Provided that the perturbation expansion over classical solutions is valid, the form of the correction is completely determined by the classical value of the entropy S_0 (a homogeneous function of second degree in the charges) and the classical values of the moduli (a homogeneous function of zero degree in the charges).

The fact that the subleading corrections are singular in S_0 drops us a hint that small black holes are purely quantum objects. In fact, the considered example of the *stu* black hole illustrates how small black hole solutions are singular in α', so that the standard perturbation expansion fails and one should find another small parameter to carry out perturbation theory (q_3 rather than $1/q_3$). Notice, however, that the inadequacy of the parameter α' as a perturbative parameter for small black holes is valid in general, not only for the considered example of the *stu* black hole. The only feature that will depend on the model is the explicit form of the genuine perturbation parameter corresponding to the small black hole limit. It would be interesting to find a model independent definition of the parameter, generalizing its specific realization (51) in the model treated in this paper.

We argued that the attractor mechanism remains valid in the presence of higher order derivative corrections, despite the fact that in the presence of flat directions in the classical black hole potential, the corrections to the entropy apparently might depend on the undefined

moduli. This is made possible by requiring, to each perturbative order in α', the fulfillment of a consistency condition, needed for the existence of a solution to a degenerate equation system, hence fixing some of the undefined moduli fields. If not all of the scalars ϕ_0 get fixed, then the group symmetry of the higher derivative corrections is a subgroup of the symmetry of the black hole potential, hence the higher order corrections will not depend on these non-fixed scalars.

We established a relation between the black hole potential approach and the entropy function formalism. The black hole potential can be deduced from the entropy function by eliminating "superfluous" fields from it. In the presence of higher derivative corrections this procedure might be quite difficult, if not impossible. Even when deriving the black hole potential proves difficult, the issue of stability of the solutions can be studied quite easily, entirely within the entropy function formalism. In this way the stability of the solutions was studied in this paper. We believe, however, that the validity of proposed method is quite general and its applicability goes beyond the considered model, and even beyond the considered class of systems – black holes in four dimensions. Hence, it would be interesting to further apply the method to study the stability also of extended objects, such as black strings and black branes, as well as rotating black objects, considered not only in four but also in higher dimensions.

In this paper we chose a specific model, i.e. the $N = 2$ $d = 4$ supergravity model with *stu* prepotential, since it is known to possess flat directions in the non-BPS branch [15]. In order to derive a form of the corrections pertinent to this theory, we started from the action of heterotic string theory [13] with all α' corrections included [7]. Consequent compactification down to four dimensions reproduces a *stu* model with all necessary corrections included. The most interesting features for us reside in this case in the sector of the axion fields; therefore we did not make any truncation over the axions.

Apart from the known solutions of the obtained model, we found two more non-BPS ones. It is noteworthy that one of these solutions acquires no α' corrections and it is a non-BPS solution with zero central charge.

We investigated the small black hole limit of the above mentioned solutions and found that the solutions found by us are singular in such a limit, while the previously known solutions become unstable in the sense that the corresponding Hessian matrix acquires positive, negative and zero eigenvalues. It is of interest to investigate whether this property of small black hole solutions is general or just a peculiarity in the considered *stu* model.

Acknowledgements

The work of S.B., S. F. and A. Y. have been supported in part by the European Research Council grant n. 226455, "SUPERSYMMETRY, QUANTUM GRAVITY AND GAUGE FIELDS (SUPERFIELDS)".

The work of A.S. has been supported by a Junior Grant of the Enrico Fermi Centre, Rome, in association with INFN Frascati National Laboratories.

A. Y. would like to thank the Department of Physics, Theory Unit Group at CERN, where this project was completed, for kind hospitality and stimulating environment.

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